

Exact Klein-Gordon equation with spatially-dependent masses for unequal scalar-vector Coulomb-like potentials

Sameer M. Ikhdaire^{1,*}

¹*Department of Physics, Near East University, Nicosia, North Cyprus, Turkey*

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Abstract

We study the effect of spatially dependent mass functions over the solution of the Klein-Gordon equation in the $(3 + 1)$ -dimensions for spinless bosonic particles where the mixed scalar-vector Coulomb-like field potentials and masses are directly proportional and inversely proportional to the distance from force center. The exact bound state energy eigenvalues and the corresponding wave functions of the Klein-Gordon equation for mixed scalar-vector and pure scalar Coulomb-like field potentials are obtained by means of the Nikiforov-Uvarov (NU) method. The energy spectrum is discussed for different scalar-vector potential mixing cases and also for constant mass case.

Keywords: Bound states, Klein-Gordon equation, position-dependent mass functions, NU method.

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*E-mail: sikhdaire@neu.edu.tr;sameer@neu.edu.tr

I. INTRODUCTION

The relativistic wave equations can be considered as first approximation to the field theory when the corrections are taken in the presence of strong potential fields [1-3]. This explains the increased interest in the Klein-Gordon (KG) and Dirac wave equations to find exact and analytic solutions for energy spectrum and wavefunctions. As a consequence of the physical importance of exact solutions of relativistic KG wave equation in quantum mechanics, under the influence of strong potentials, an increasing interest in this equation has appeared in the study of quark-antiquark mass spectroscopy [4-6], atomic, nuclear, and plasma physics [7-9]. So, the idea is to use the bosonic particle KG equation as a mathematical tool to reach the goal of obtaining approximate solutions for nonrelativistic fermionic particles eigenstates. The problems that can be solved exactly in relativistic quantum mechanics are very limited perhaps because of the mathematical difficulties.

Recently, the bound and scattering solutions of the s - and l -waves KG and Dirac equations for any interaction system have raised a great interest [10-12]. The bound-states of the Dirac and KG equations with the Coulomb-like scalar plus vector potentials have been studied in arbitrary dimension [13-17]. Furthermore, the exact results for the scattering states of the KG equation with Coulomb-like scalar plus vector potentials have been investigated in an arbitrary dimension [18]. This equation has been exactly solved for a larger class of linear, exponential and linear plus Coulomb potentials to determine the bound state energy spectrum using two semiclassical methods [19]. Many authors have considered a more general transformation between the unequal vector and scalar potentials given by

$$V(r) = V_0 + \beta S(r), \quad (1)$$

where V_0 and β being arbitrary constants of certain proportions have to be chosen after solving the problem under consideration [19-21]. It is interesting to note that, this restriction includes the case where $V(r) = 0$, when both constants vanish, the situation where the potentials are equal in magnitude and sign $V(r) = S(r)$ or equal in magnitude but opposite in sign $V(r) = -S(r)$ (*i.e.*, $V_0 = 0; \beta = \pm 1$), and also the case where the potentials are proportional when $V_0 \neq 0$ [21]. For the case when $S(r) \geq V(r)$, there exist bound state solutions. However, it should be noticed that the case when the scalar potential is equal to the vector potential must be considered separately [3]. Under the condition of $S(r) = V(r)$, the KG turns into a Schrödinger-like equation and thus the bound state solutions are very

easily to obtain with the help of the well-known methods developed in the non-relativistic quantum mechanics.

On the other hand, the problem of the spatially-dependent effective mass is presenting a growing interest along the last few years [22-25]. Many authors have used different methods to study the partially exactly solvable and exactly solvable Schrödinger, KG and Dirac equations in the presence of variable mass having a suitable mass distribution functions in $1D$, $3D$ and/or any D -dimensional cases for different potentials, such as the linear potential [20], the exponential-type potentials [21], the Coulomb potential [26], the Lorentz scalar interactions [27], the hyperbolic-type potentials [28], the Morse potential [29], the Pöschl-Teller potential [30], the inversely linear scalar potential [31], the Coulomb and harmonic potentials [32], the modified Kratzer-type, rotationally corrected Morse potentials [33], Mie-type and pseudoharmonic potentials [34]. Recently, the point canonical transformation (PCT) has also been employed to solve the D -dimensional spatially dependent mass Schrödinger equation for some molecular potentials to get the exact bound state solutions including the energy spectrum and corresponding wave functions [32-34]. It is quite natural to look for relativistic treatment of this type of systems, mostly because the ordering ambiguity which is present in the nonrelativistic case [35], is expected to be avoided under relativistic ambiance [36,37].

Very recently, the NU method has been used to solve any l -states KG equation approximately for the Hulthén potential with a suitable choice of spatially-dependent mass function distribution of an exponential-type [38]. Also, a new approximation scheme [39] has been proposed for the centrifugal term to obtain a quasi-exact analytic bound-state solution of the radial KG equation with spatially-dependent effective mass for scalar and vector Hulthén potentials in any arbitrary dimension and orbital angular momentum quantum number l within the framework of the NU method [40].

The problem of a particle subject to an inversely linear potential in one spatial dimension ($\sim |x|^{-1}$), known as the one-dimensional hydrogen atom, by considering a convenient mixing of vector and scalar Lorentz structures has received considerable attention in the literature [41,42]. The same problem for a spinless particle subject to a general mixing of vector and scalar inversely linear potentials in the (1+1)-dimensional world was analyzed [43]. Exact bounded solutions were found in closed form by imposing boundary conditions on the eigenfunctions which ensure that the effective Hamiltonian is Hermitian for all the points of the space. Jia and Souza Dutra [44] considered position-dependent effective mass Dirac

equations with PT and non- PT symmetric potentials. Souza Dutra and Jia [31], investigated the exact solution of the (1+1)-dimensional KG equation with spatially dependent mass for the inversely linear potential. Recently, in ref. [45] the bound state solutions of the (1+1)-dimensional KG equation with mass inversely proportional to the distance from the force center for the inversely linear potential were obtained by using the NU method [46-52]. Two particular cases are studied, the case when vector potential is equal to the scalar potential in magnitude $V(r) = S(r)$ and when vector potential is equal to the scalar potential in magnitude but not in sign $V(r) = -S(r)$, (*i.e.*, $V_0 = 0; \beta = \pm 1$).

In the present work, we feel tempted to extend the work of ref. [45] to study the bound state solutions of the (3 + 1)-dimensional KG equation with position-dependent bosonic mass function $m(r) = m_0(1 + \lambda_0 br^{-1})$ where $r \neq 0$ [45] for the attractive scalar potential $S(r) = -\kappa_s r^{-1}$ with $\kappa_s = \hbar c q_s$ being the coupling constant, taking into consideration the general mixings of scalar and vector Lorentz structure potential given in eq. (1). Firstly, this choice of mass function together with potential mixings is mostly suitable for modeling some physical systems like the Kratzer-type potentials [48]. Secondly, the motivation for this choice is due to the nature of the dominating Coulombic field between the two interacting nuclei at short distances. Thirdly, this choice enables one to solve the KG equation easily and elegantly. The general mixings of potentials include: (i) $V(r) = -\kappa_v r^{-1}$ and $S(r) = 0$, which represents a π^- meson in a Coulomb field. (ii) $V(r) = 0$ and $S(r) = -\kappa_s r^{-1}$, which has no experimental evidence. (iii) $V(r) = S(r) = -\kappa r^{-1}$, ($\kappa_s = \kappa_v = \kappa$, $V_0 = 0$, $\beta = 1$) which represents not only a KG particle in an equally mixed Lorentz scalar and Lorentz vector potentials but also a Dirac particle in the same potential mixture, where $l = j + 1/2$ and the radial KG wave function represents the radial large-component of the Dirac spinor [53]. (iv) $V(r) = -\kappa_v r^{-1} + Ar$ and $S(r) = 0$ representing a π^- meson in a Coulomb field perturbed by a linear Lorentz vector interaction Ar . Also, we consider the effect of a spatially dependent mass of the linear form $m(r) = m_0 r/L$ [45] on the solution of the (3+1)-dimensional KG equation for the Lorentz vector and scalar potentials of the form $V(r) = 0$ and $S(r) = sr^{-1}$, respectively. It is worth mentioning that this choice of mass function together with potential mixings is mostly suitable for modeling some physical systems like the Pseudoharmonic potential.

The paper is organized as follows. In sect. 2, we outline the NU method. Section 3 is devoted for the bound state analytic solutions of the (3 + 1)-dimensional KG equation with

spatially dependent mass functions for two quantum systems obtained by means of the NU method. Finally, the relevant results are discussed in sect. 4.

II. NU METHOD

The NU method is briefly outlined here and the details can be found in ref. [46]. This method is proposed to solve the second-order differential equation of the hypergeometric type:

$$\psi_n''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi_n'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi_n(z) = 0, \quad (2)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials, at most, of second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. In order to find a particular solution for eq. (2), let us decompose the wavefunction $\psi_n(z)$ as follows:

$$\psi_n(z) = \phi_n(z)y_n(z), \quad (3)$$

and use

$$[\sigma(z)\rho(z)]' = \tau(z)\rho(z), \quad (4)$$

to reduce eq. (2) to the form

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0, \quad (5)$$

with

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z), \quad \tau'(z) < 0, \quad (6)$$

where the prime denotes the differentiation with respect to z . One is looking for a family of solutions corresponding to

$$\lambda = \lambda_n = -n\tau'(z) - \frac{1}{2}n(n-1)\sigma''(z), \quad n = 0, 1, 2, \dots, \quad (7)$$

The $y_n(z)$ can be expressed in terms of the Rodrigues relation:

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n} [\sigma^n(z)\rho(z)], \quad (8)$$

where B_n is the normalization constant and the weight function $\rho(z)$ is the solution of the differential equation (4). The other part of the wavefunction (3) must satisfy the following logarithmic equation

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)}. \quad (9)$$

By defining

$$k = \lambda - \pi'(z). \quad (10)$$

one obtains the polynomial

$$\pi(z) = \frac{1}{2} [\sigma'(z) - \tilde{\tau}(z)] \pm \sqrt{\frac{1}{4} [\sigma'(z) - \tilde{\tau}(z)]^2 - \tilde{\sigma}(z) + k\sigma(z)}, \quad (11)$$

where $\pi(z)$ is a parameter at most of order 1. The expression under the square root sign in the above equation can be arranged as a polynomial of second order where its discriminant is zero. In this regard, an equation for k is being obtained. After solving such an equation, the k values are determined through the NU method.

III. EXACT BOUND-STATE SOLUTIONS

In the relativistic quantum mechanics, for a spinless particle, we write the full stationary KG equation for a spatially dependent bosonic mass in real $(3+1)$ -dimensions as [49,50]

$$\nabla^2 \psi(\mathbf{r}) + \frac{1}{\hbar^2 c^2} \left\{ [E_{nl} - V(r)]^2 - [m(r)c^2 + S(r)]^2 \right\} \psi(\mathbf{r}) = 0, \quad \nabla^2 = \sum_{j=1}^3 \frac{\partial^2}{\partial x_j^2} \quad (12)$$

where $m(r)$ is a bosonic mass, E_{nl} is the energy of the particle, $V(r)$ is a Lorentz vector (coupled as the 0-component of the four-vector potential) and $S(r)$ is a Lorentz scalar (added to the mass term) potentials. Let us decompose the radial wavefunction $\psi(\mathbf{r})$ as follows:

$$\psi(\mathbf{r}) = \frac{u(r)}{r} Y_m^{(l)}(\hat{\mathbf{r}}), \quad (13)$$

where $u(r)$ is the radial wave function and $Y_m^{(l)}(\hat{\mathbf{r}})$ is the angular dependent spherical harmonics and this reduces eq. (12) into the following position-dependent effective mass Schrödinger-like equation:

$$\begin{aligned} \frac{d^2 u(r)}{dr^2} + \frac{1}{\hbar^2 c^2} \left\{ E_{nl}^2 + V(r)^2 - 2EV(r) - m(r)^2 c^4 - S(r)^2 - 2m(r)c^2 S(r) - \frac{l(l+1)\hbar^2 c^2}{r^2} \right\} \\ \times u(r) = 0, \quad u(0) = 0. \end{aligned} \quad (14)$$

Now, we will start to analyze some illustrative particular cases for Lorentz scalar-vector mixings and suitable mass distribution functions for modeling some important physical systems.

A. Mixed vector-scalar Coulomb potentials

We shall present a general solution of ref. [46] for general admixture of scalar and vector potential mixings. Let us solve eq. (14) for the general relationship between vector and scalar potentials given in eq. (1), then we have

$$\begin{aligned} -\frac{d^2 u(r)}{dr^2} + \frac{1}{\hbar^2 c^2} \left[(1 - \beta^2) S(r)^2 - 2((E_{nl} + V_0)\beta + m(r)) S(r) + \frac{l(l+1)\hbar^2 c^2}{r^2} \right] u(r) \\ = \frac{1}{\hbar^2 c^2} [(E_{nl} + V_0)^2 - m^2(r)c^4] u(r). \end{aligned} \quad (15)$$

Furthermore, we take the scalar potential in the form of an attractive Coulomb-like field

$$S(r) = -\frac{\hbar c q_s}{r}, \quad q_s = q, \quad r \neq 0 \quad (16)$$

where q_s is being a scalar dimensionless real parameter coupling constant and $\hbar c$ is being a constant with $J.f m$ dimension. At this stage, it is worthwhile to mention that the above choice of the mass function of Coulombic form together with the presently taken admixture of scalar and vector fields are mostly suitable for modeling the well-known pseudo-Coulomb (Kratzer-type) potentials [48]. Following refs. [38,40,45], we may take the spatially-dependent mass function being described by

$$m(r) = m_0 \left(1 + \frac{\lambda_0 b}{r} \right), \quad r \neq 0, \quad (17)$$

where m_0 and $\lambda_0 = \frac{\hbar}{m_0 c}$ are the rest mass of the bosonic particle and the Compton-like wavelength in $f m$ units, respectively; b is a dimensionless real constant. The interaction field has much impact on the choice of the mass function which, in the present case, is inversely proportional to the distance between the two nuclei at short distances $m(r) \sim \frac{1}{r}$ and constant at long distances $m(r \rightarrow \infty) \simeq m_0$. Let us introduce the variable change $z = r \in (0, \infty)$ and define

$$\begin{aligned} \varepsilon_{nl} = \frac{\sqrt{m_0^2 c^4 - \tilde{E}_{nl}^2}}{Q} \quad (m_0 c^2 \geq \tilde{E}_{nl}), \quad \gamma_1 = \frac{2(b-q)m_0 c^2 - 2q\beta\tilde{E}_{nl}}{Q}, \\ \tilde{E}_{nl} = E_{nl} + V_0, \quad \gamma_2 = b(b-2q) + q^2(1-\beta^2) + l(l+1), \quad Q = \hbar c, \end{aligned} \quad (18)$$

with the following constraint $V_0 \leq -E_{nl} + m_0 c^2$ must be fulfilled for bound state solutions. Further, substituting eqs. (16)-(18) into eq. (15), we obtain

$$\frac{d^2 u(z)}{dz^2} - \left(\frac{\varepsilon_{nl}^2 z^2 + \gamma_1 z + \gamma_2}{z^2} \right) u(z) = 0, \quad u(0) = 0, \quad (19)$$

where $u(z) = u(r)$. In the present work, we deal with bound state solutions, the quantum condition is obtained from the finiteness of the solution at infinity, *i.e.*, the wave function $u(r)$ must satisfy boundary conditions, $u(r) = 0$ when $r \rightarrow \infty$ and at the origin point, $r = 0$. In order to solve eq. (19) by means of the NU method, we should compare it with eq. (2). The following values for parameters are found:

$$\tilde{\tau}(z) = 0, \quad \sigma(z) = z, \quad \tilde{\sigma}(z) = -(\varepsilon_{nl}^2 z^2 + \gamma_1 z + \gamma_2). \quad (20)$$

Inserting these values of parameters into eq. (11), we obtain

$$\pi(z) = \frac{1}{2} \pm \frac{1}{2} \sqrt{4\varepsilon_{nl}^2 z^2 + 4(\gamma_1 + k)z + 4\gamma_2 + 1}. \quad (21)$$

The discriminant of the square root must be set equal to zero, *i.e.*, $\Delta = 4\varepsilon_{nl}^2 z^2 + 4(\gamma_1 + k)z + 4\gamma_2 + 1 = 0$. Consequently, the following two constants k_1 and k_2 are found to be

$$k_{1,2} = -\gamma_1 \pm \varepsilon_{nl} \sqrt{1 + 4\gamma_2}, \quad (22)$$

with the following requirements on the parameters $\beta \leq 2\sqrt{(l + \frac{1}{2})^2 + (q - b)^2}$ and $V_0 \leq -E_{nl} + m_0 c^2$ must be fulfilled for real solutions. In this regard, we can find the possible functions for $\pi(z)$ as

$$\pi(z) = \begin{cases} \frac{1}{2} \pm [\varepsilon_{nl} z + \frac{1}{2} \sqrt{1 + 4\gamma_2}] & \text{for } k_1 = -\gamma_1 + \varepsilon_{nl} \sqrt{1 + 4\gamma_2}, \\ \frac{1}{2} \pm [\varepsilon_{nl} z - \frac{1}{2} \sqrt{1 + 4\gamma_2}] & \text{for } k_2 = -\gamma_1 - \varepsilon_{nl} \sqrt{1 + 4\gamma_2}. \end{cases} \quad (23)$$

According to the NU method, one of the four values of the polynomial $\pi(z)$ is just proper to obtain the energy states because $\tau(z)$ has a negative derivative for this value of $\pi(z)$. Therefore, the selected forms of $\pi(z)$ and k take the following particular values

$$\pi(z) = -\varepsilon_{nl} z + \frac{1}{2} \left(1 + \sqrt{1 + 4\gamma_2} \right), \quad k = -\gamma_1 - \varepsilon_{nl} \sqrt{1 + 4\gamma_2}, \quad (24)$$

to obtain

$$\tau(z) = -2\varepsilon_{nl} z + 1 + \sqrt{1 + 4\gamma_2}, \quad \tau'(z) = -2\varepsilon_{nl} < 0, \quad (25)$$

where $\tau'(z) = \frac{d\tau(z)}{dz}$. In addition, after using eqs. (24) and (25) together with the assignments given in eq. (20), the following expressions for λ are obtained

$$\lambda_n = \lambda = 2n\varepsilon_{nl}, \quad n = 0, 1, 2, \dots, \quad (26)$$

$$\lambda = -\gamma_1 - \varepsilon_{nl} \left(1 + \sqrt{1 + 4\gamma_2} \right). \quad (27)$$

Letting $\lambda_n = \lambda$, we can solve the above equations for the energy states E_{nl}^\pm as

$$E_{nl}^\pm = -V_0 + \frac{\left[q(b-q)\beta \pm B_{nl}\sqrt{B_{nl}^2 - q^2(1-\beta^2) - b(b-2q)} \right]}{q^2\beta^2 + B_{nl}^2} m_0 c^2, \quad (28)$$

where

$$B_{nl} = n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2}\right)^2 + b(b-2q) + q^2(1-\beta^2)}, \quad n, l = 0, 1, 2, \dots \quad (29)$$

For spatially-dependent mass case, *i.e.*, $b \neq 0$, the bound state solutions of the system are determined by the parameters q and b . It is not difficult to conclude that all bound-states appear in pairs, two energy solutions are valid for the particle $E^p = E_{nl}^+$ and the second one corresponds to the anti-particle energy $E^a = E_{nl}^-$ in the Coulomb-like field.

Let us now find the corresponding eigenfunctions for this system. Using eqs. (4) and (9), we find

$$\rho(z) = z^{\sqrt{1+4\gamma_2}} e^{-2\varepsilon_{nl}z}, \quad (30)$$

$$\phi(z) = z^{(1+\sqrt{1+4\gamma_2})/2} e^{-\varepsilon_{nl}z}. \quad (31)$$

Hence, substituting eq. (30) into eq. (8), we find

$$y_n(z) = D_n z^{-\sqrt{1+4\gamma_2}} e^{2\varepsilon_{nl}z} \frac{d^n}{dz^n} \left[z^{(n+\sqrt{1+4\gamma_2})} e^{-2\varepsilon_{nl}z} \right] \sim L_n^{2L+1}(2\varepsilon_{nl}z), \quad (32)$$

where $L_n^\alpha(x)$ is the generalized Laguerre polynomials. By using $u(z) = \phi(z)y_n(z)$, we get the wavefunctions as

$$u(r) = N r^{(1+\sqrt{1+4\gamma_2})/2} e^{-\varepsilon_{nl}r} L_n^{2L+1}(2\varepsilon_{nl}r), \quad (33)$$

where

$$L = \sqrt{\left(l + \frac{1}{2}\right)^2 + b(b-2q) + q^2(1-\beta^2)} - \frac{1}{2}. \quad (34)$$

Using the normalization condition $\int_0^\infty u(r)^2 dr = 1$ and the orthogonality relation of the generalized Laguerre polynomials $\int_0^\infty x^{\alpha+1} e^{-x} \left[L_n^{(\alpha)}(x) \right]^2 dx = (2n + \alpha + 1) \frac{\Gamma(n+\alpha+1)}{n!}$, the normalizing factor N can be found as [54-57]

$$N = \sqrt{\frac{n! (2\varepsilon_{nl})^{2L+3}}{2(n+L+1)\Gamma(n+2L+2)}}, \quad (35)$$

where ε_{nl} and L are given in eqs. (18) and (34), respectively.

(1) If we consider the case when scalar potential is equal the vector potential in magnitude and sign, *i.e.*, $V_0 = 0$ and $\beta = 1$, then we have

$$E_{nl}^p = \frac{\left[q(b-q) + B_{nl} \sqrt{B_{nl}^2 - b(b-2q)} \right]}{q^2 + B_{nl}^2} m_0 c^2, \quad (36a)$$

$$E_{nl}^a = \frac{\left[q(b-q) - B_{nl} \sqrt{B_{nl}^2 - b(b-2q)} \right]}{q^2 + B_{nl}^2} m_0 c^2, \quad (36b)$$

where

$$B_{nl} = n + \frac{1}{2} + \sqrt{\left(l + \frac{1}{2} \right)^2 + b(b-2q)}, \quad n, l = 0, 1, 2, \dots \quad (37)$$

Obviously, the bound state solutions of the particle and anti-particle are available. When the mass is taken to be constant, *i.e.*, $b = 0$, we have

$$E_{nl}^p = \frac{(n+l+1)^2 - q^2}{(n+l+1)^2 + q^2} m_0 c^2, \quad E_{nl}^a = -m_0 c^2, \quad n, l = 0, 1, 2, \dots, \quad (38)$$

where n and l signify the usual radial and orbital quantum numbers. The particle has bound state solution whereas anti-particle has continuum solution for all states.

(2) If we consider the case when scalar potential is equal to the vector potential in magnitude but not in sign, *i.e.*, $V_0 = 0$ and $\beta = -1$, then we have

$$E_{nl}^p = \frac{\left[-q(b-q) + B_{nl} \sqrt{B_{nl}^2 - b(b-2q)} \right]}{q^2 + B_{nl}^2} m_0 c^2, \quad (39a)$$

$$E_{nl}^a = \frac{\left[-q(b-q) - B_{nl} \sqrt{B_{nl}^2 - b(b-2q)} \right]}{q^2 + B_{nl}^2} m_0 c^2. \quad (39b)$$

For the constant-mass case, *i.e.*, $b = 0$, we have

$$E_{nl}^p = m_0 c^2, \quad E_{nl}^a = -\frac{(n+l+1)^2 - q^2}{(n+l+1)^2 + q^2} m_0 c^2. \quad (40)$$

Obviously, the particle has continuum solution for all states whereas bound state solution for anti-particle. In addition, when the potential coupling constant is taken as $q = b/2$, the spectra of the varying mass KG particle in potential fields $q_s = q_v$ ($q_s = -q_v$) are similar to the spectra of constant mass KG particle in the potential fields $q_s = -q_v$ ($q_s = q_v$), respectively.

B. Pure Scalar Coulomb-like potential

In their paper [44], Souza Dutra and Jia used the pure scalar potential that is inversely proportional to the absolute value of the coordinate. Here, we use a pure scalar repulsive Coulomb-like field potential

$$S(r) = \frac{s}{r}, \quad V(r) = 0, \quad (41)$$

with s being a coupling parameter with $J.f m$ dimension and also assume the spatially-dependent mass function having a linear form

$$m(r) = Ar, \quad (42)$$

with $A = \frac{m_0}{L}$ where m_0 is being the rest mass and L is being a constant with space dimension. At this stage, it is worthwhile to mention that the above choice of the mass function together with the presently taken pure scalar potential case are mostly suitable for modeling the well-known pseudoharmonic potentials [58,59]. Inserting eqs. (41) and (42) into eq. (15), then we have

$$-\frac{d^2 u(r)}{dr^2} + \frac{1}{\hbar^2 c^2} \left[\frac{m_0^2 c^4}{L^2} r^2 + \frac{s^2 + l(l+1)\hbar^2 c^2}{r^2} \right] u(r) = \frac{1}{\hbar^2 c^2} \left(E_{nl}^2 - \frac{2m_0 c^2 s}{L} \right) u(r). \quad (43)$$

Thus, the present problem has been reduced to three-dimensional Schrödinger equation for pseudoharmonic oscillator problem which was solved before in refs. [58,59]. Introducing the variable change $z = r^2 \in (0, \infty)$ and defining

$$\varepsilon_{nl} = \frac{1}{\hbar c} \sqrt{\frac{2m_0 c^2 s}{L} - E_{nl}^2}, \quad \alpha_1 = \frac{m_0 c}{\hbar L}, \quad \alpha_2 = \frac{s^2 + l(l+1)\hbar^2 c^2}{\hbar^2 c^2}, \quad (44)$$

we obtain

$$\frac{d^2 u(z)}{dz^2} + \frac{1}{2z} \frac{du(z)}{dz} + \frac{1}{(2z)^2} (-\alpha_1^2 z^2 - \varepsilon_{nl}^2 z - \alpha_2) u(z) = 0, \quad (45)$$

where $u(z) = u(r)$. Comparing eq. (45) with eq. (2), we find values for the parameters as

$$\tilde{\tau}(z) = 1, \quad \sigma(z) = 2z, \quad \tilde{\sigma}(z) = -\alpha_1^2 z^2 - \varepsilon_{nl}^2 z - \alpha_2, \quad (46)$$

and by inserting these values of parameters into eq. (11), we further obtain

$$\pi(z) = \frac{1}{2} \pm \frac{1}{2} \sqrt{4\alpha_1^2 z^2 + 4(\varepsilon_{nl}^2 + k)z + 4\alpha_2 + 1}, \quad (47)$$

and the constant k as

$$k_{1,2} = -\varepsilon_{nl}^2 \pm \alpha_1 \sqrt{4\alpha_2 + 1}. \quad (48)$$

When the individual values of k given in eq. (48) are being substituted into Eq. (47), the four possible forms of $\pi(z)$ are written as follows

$$\pi(z) = \begin{cases} \frac{1}{2} \pm \frac{1}{2} [2\alpha_1 z + \sqrt{4\alpha_2 + 1}] & \text{for } k_1 = -\varepsilon_{nl}^2 + \alpha_1 \sqrt{4\alpha_2 + 1}, \\ \frac{1}{2} \pm \frac{1}{2} [2\alpha_1 z - \sqrt{4\alpha_2 + 1}] & \text{for } k_2 = -\varepsilon_{nl}^2 - \alpha_1 \sqrt{4\alpha_2 + 1}. \end{cases} \quad (49)$$

According to the NU method, the selected forms of $\pi(z)$ and k are taking the following particular values

$$\pi(z) = -\alpha_1 z + \frac{1}{2} [1 + \sqrt{4\alpha_2 + 1}], \quad k = -\varepsilon_{nl}^2 - \alpha_1 \sqrt{4\alpha_2 + 1}, \quad (50)$$

to obtain

$$\tau(z) = -2\alpha_1 z + 2 + \sqrt{4\alpha_2 + 1}, \quad \tau'(z) = -2\alpha_1 < 0, \quad (51)$$

which is the essential condition in the method. Also, the following expressions for λ are obtained

$$\lambda_n = \lambda = 2n\alpha_1, \quad n = 0, 1, 2, \dots, \quad (52)$$

$$\lambda = -\varepsilon_{nl}^2 - \alpha_1 (1 + \sqrt{4\alpha_2 + 1}). \quad (53)$$

Letting $\lambda_n = \lambda$, we can solve the above equations for the energy eigenvalues E_{nl} as

$$\tilde{E}_{nl} = \alpha_1 (1 - 2A_{nl}), \quad (54)$$

where

$$\tilde{E}_{nl} = -\varepsilon_{nl}^2, \quad A_{nl} = - \left(n + \frac{1}{2} \sqrt{(2l+1)^2 + \frac{4s^2}{\hbar^2 c^2}} \right), \quad (55)$$

and thus we find

$$\frac{E_{nl}^2}{m_0 c^2} = \frac{2s}{L} + \frac{\hbar c}{L} \left(2n + 1 + \sqrt{(2l+1)^2 + \frac{4s^2}{\hbar^2 c^2}} \right), \quad (56)$$

which is found to be consistent with eq. (20) of ref. [51] obtained by SUSY method when l is set equal to zero. We note that the energy levels for particles and antiparticles are symmetric about $E_{nl} = 0$ [49].

Essentially, we should report that eq. (43) corresponds to the Schrödinger equation of anharmonic oscillator potential $V(r) = \alpha_1^2 r^2$, with energy levels [58,59]

$$\tilde{E}_{nl} = \alpha_1 (2n + 2\Lambda + 3), \quad (57)$$

with \tilde{E}_{nl} is given in (55), α_1 in (44) and Λ is defined by

$$\Lambda = \frac{1}{2} \left(\sqrt{(2l+1)^2 + \left(\frac{2s}{\hbar c}\right)^2} - 1 \right). \quad (58)$$

Let us now find the corresponding eigenfunctions for this system. After using eqs. (4) and (9), we find

$$\rho(z) = z^{\sqrt{4\alpha_2+1}/2} e^{-\alpha_1 z}, \quad (59)$$

$$\phi(z) = z^{(1+\sqrt{4\alpha_2+1})/4} e^{-\alpha_1 z/2}. \quad (60)$$

Substituting eq. (59) into eq. (8), we obtain

$$y_n(z) = D_n z^{-\sqrt{4\alpha_2+1}/2} e^{\alpha_1 z} \frac{d^n}{dz^n} \left[z^{(n+\sqrt{4\alpha_2+1}/2)} e^{-\alpha_1 z} \right] \sim L_n^{(2\Lambda+1)/2}(\alpha_1 z). \quad (61)$$

By using $u(z) = \phi(z)y_n(z)$, we get the wavefunctions as

$$u(r) = N e^{-\alpha_1 r^2/2} r^{(\Lambda+1)/2} L_n^{(2\Lambda+1)/2}(\alpha_1 r^2), \quad (62)$$

where Λ is defined in eq. (58). It is worth mentioning that the above wave function is consistent with eq. (20) of ref. [58] in the solution of the Schrödinger equation for the pseudoharmonic oscillator potential. Essentially, such a solution has been discussed before by many authors [59]. Making use of the normalization condition $\int_0^\infty u(r)^2 dr = 1$ and the orthogonality relation of the generalized Laguerre polynomials $\int_0^\infty x^{\alpha'-1} e^{-x} \left[L_n^{(\alpha)}(x) \right]^2 dx = \binom{\alpha - \alpha' + n}{n} \Gamma(\alpha')$, the normalization constant N can be found as [54-57]

$$N = \sqrt{\frac{2 \left(\frac{m_0 c}{\hbar L}\right)^{\frac{1}{2}} \sqrt{(2l+1)^2 + \left(\frac{2s}{\hbar c}\right)^2} + 1}{\binom{n-1}{n} \Gamma\left(\frac{1}{2} \sqrt{(2l+1)^2 + \left(\frac{2s}{\hbar c}\right)^2} + 1\right)}}. \quad (63)$$

IV. CONCLUSIONS

We have extended the (1+1)-dimensional KG solution in ref. [45] to the l -waves KG for scalar-vector mixing Coulomb-like fields with suitable choices of spatially dependent mass functions. Thus, for this kind of studied problems, we may conclude that the relativistic wave equation can be solved exactly. For suitable choices of potential forms as the general

mixing of scalar-vector and pure scalar Coulomb-like field potential, the relativistic bound state energy spectrum and wave functions have been obtained, respectively. The resulting solutions of the wave functions are being expressed in terms of the generalized Laguerre polynomials. We have considered different mass functions of inversely proportional and directly proportional to the coordinate distance. Obviously, when the coupling potential parameters are adjusted to some specific values, particularly when $q = b/2$, the spectra of the mass varying KG particle for the case $q_s = q_v$ ($q_s = -q_v$) become similar to the spectra of the constant mass KG particle for the case $q_s = -q_v$ ($q_s = q_v$), respectively. It is found that the KG equation with a suitable mass function for a pure scalar potential is being reduced into the constant mass Schrödinger equation for the anharmonic oscillator potential. In the limit of constant mass ($b = 0$), the solution for the energy eigenvalues and wave functions are reduced to those ones in literature. Also, when $l = 0$, the problem reduces to s -waves solution.

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